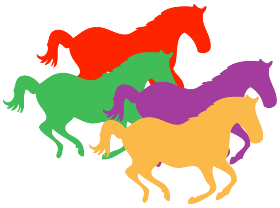
MATHS CHALLENGE 2010
SOLUTIONS – PART 3

HARD 1

Four horses enter the prestigious MathsMasters Cup. Assuming that there are no dead heats, how many different ways are there for the horses to cross the finish line? What if we include the possibility of dead heats?



ANSWER: Assuming no dead heats, there are 4 x 3 x 2 x 1 = 24 orders in which the horses can finish. If there are dead heats, we have to consider various possibilities:

a) No horses dead heat. Then, as indicated, there are 24 possible finishes.
b) Two horses dead heat. There are 6 choices of the two horses that tie, and then 3 x 2 x 1 = 6 orderings for the three finishers (considering the tied pair as one finisher). That gives 6 x 6 = 36 possible finishes.
c) Two pairs of horses separately dead heat. There are 3 ways of separating the four horses into two pairs, and then 2 ways in which two pairs might finish. So, there are 3 x 2 = 6 possible finishes.
d) Three horses dead heat. There are 4 ways to choose the tying horses, and then 2 ways to choose the ordering, giving a total of 4 x 2 = 8 possible finishes.
e) All four horses dead heat. There is 1 way to do this.

Adding up the possibilities, there are 24 + 36 + 6 + 8 + 1 = 75 ways the horses can finish.

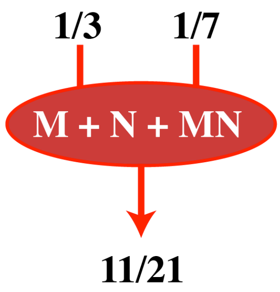
HARD 2

The fictional blockbuster *Tropic of Calculus* has just been published. Its pages are numbered starting with page 1, and there are 2893 digits used to number all the pages. How many pages does the book have?

ANSWER: The first 9 pages use 9 digits. The next 90 pages (from 10 to 99), use 90 x 2 = 180 digits. Then, the next 900 pages, up to page 999, use 900 x 3 = 2700 digits. That is a total of 2889 digits. Page 1000 takes the total digits to 2893, so the book is 1000 pages long.

HARD 3

Start with the numbers 1, 1/2, 1/3, … , 1/100. Choose any two numbers *M* and *N* from this list and replace them by the single number *M* + *N* + *M* x *N*. Repeat the process until only one number is left. What is that number?



ANSWER: If we perform the operation on the first two numbers, 1 and 1/2, the result is 2. If we then perform the operation on 2 and the third number, 1/3, the result is 3. Continuing the operating in this order leaves a final number 100.

What is less easy to see is that the final number is the same, no matter in which order we choose to combine the numbers. Suppose we start with three numbers, *M*, *N* and *Q*. Combining *M* and *N*, and then *Q* gives

$$(M + N + M \times N) + Q + (M + N + M \times N) \times Q$$

$$= \quad M + N + Q + M \times N + M \times Q + N \times Q + M \times N \times Q$$

So, the result is symmetric in *M*, *N* and *Q*, meaning the order of combination wouldn't have mattered. Repeating the argument (by "mathematical induction" for the pedants), the result of combining our hundred fractions is the same, no matter the order in which we choose them.

HARD 4

What is the value of the following root monster?

$$\sqrt{22+\sqrt[3]{22+\sqrt{22+\sqrt[3]{22+\sqrt{22+\sqrt[3]{22+\sqrt{22+\cdots}}}}}}}$$

ANSWER: Call the value of the monster *X*. Then squaring gives X^2 = 22 + *cube root thingo*. Subtracting 22 from both sides and cubing, we find

$$\left(X^2 - 22\right)^3 = 22 + X$$

We can see that *X* = 5 is a solution to this equation, and it is pretty easy to check that there are no other real solutions.

HARD 5

Burkard and Marty walk up a moving escalator. They start together on the bottom step of the escalator, with Burkard taking two steps for each step that Marty takes. Burkard steps off the escalator on his 28th step, and Marty does so on his 21st step. How many steps of the escalator are visible at any one time?

ANSWER: Suppose there are X steps visible on the elevator, and that Y steps move by as Burkard walks. Then $28 + Y = X$. Marty's total steps take $2 \times (21/28) = 3/2$ as long as Burkard's, and so $3Y/2$ steps will move by as Marty walks. That means $21 + 3Y/2 = X$. Subtracting the second equation from the first, we find $7 = Y/2$. So, $Y = 14$, and there are $X = 42$ steps.

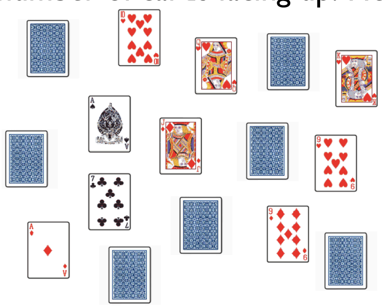
HARD 6

At the MathsSnacks fast food outlet, you can order Kitty Nibbles in boxes of 6, 9 and 20. If you wanted exactly 25 Kitty Nibbles, for example, then no selection of boxes will satisfy you. What is the largest number of Kitty Nibbles such that no selection of boxes will give that exact number?

ANSWER: Suppose we want a total of N Kitty Nibbles. If $N > 3$ and N is a multiple of 3 then we can easily use a combination of the 6-boxes and the 9-boxes to obtain the desired N nibbles. Next, if $N > 43$ then either N or $N - 20$ or $N - 40$ is a multiple of three. So, by maybe using one or two 20-boxes as well, any number $N > 43$ can be obtained exactly. Finally, it is easily seen that exactly 43 Kitty Nibbles cannot be obtained.

HARD 7

There are 100 playing cards on the tabletop, and exactly 23 cards are face up. Blindfolded, you want to separate the cards into two groups, each with the same number of cards facing up. How do you do it?

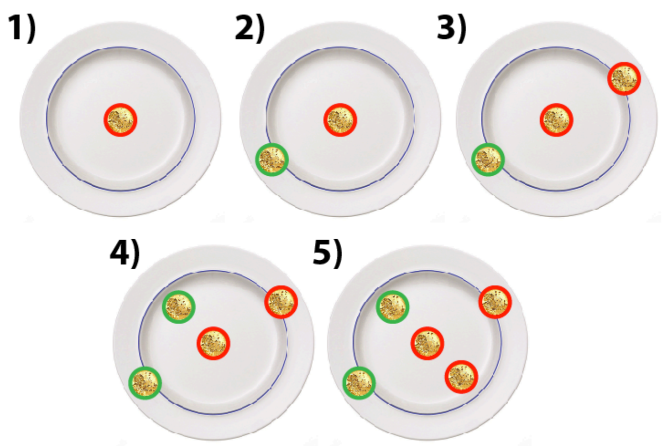


ANSWER: Ensure one group contains exactly 23 cards. If *N* of the cards in this group are face up, then 23 – *N* of the cards in the other group are face up. Now, take the group of 23 cards and flip them all over. Now, in both groups exactly 23 – *N* of the cards are face up.

HARD 8

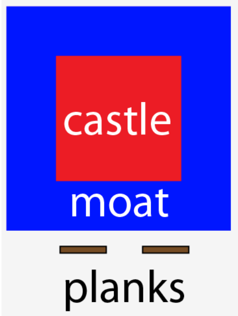
Ted and John play a game of placing $2 coins on a flat plate. The coins cannot overlap, and the winner is the person to place the last coin to fit on the plate. Ted goes first. Who has a winning strategy, and what is it?

ANSWER: Ted wins by placing his first coin in the centre. After that, wherever John places his coin, Ted plays in the symmetrically opposite position.

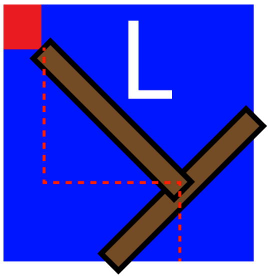


HARD 9

A castle is surrounded by a moat which is 20 meters wide. The brave Sir Isaac Newton wants to enter the castle, to rescue the fair maiden. He has only two planks, both of length 19 meters, and no nails. How can Sir Newton cross the moat?



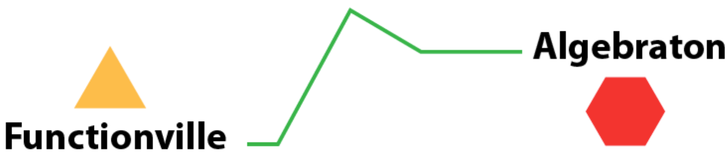
ANSWER: Use one of the planks to cut off the corner of the moat, as pictured. Then use the second plank to reach from the corner to the castle.



Suppose the planks are of length L, Pythagoras shows that the moat width Sir Isaac can cross is $L/\sqrt{2} + (L/2)/\sqrt{2}$. Since our planks are of length $L = 19$, Sir Isaac can cross a moat of width $(57\sqrt{2})/4$, which is just greater than 20 meters.

HARD 10

Emmy Noether's car travels 120 kph downhill, 80 kph on level ground and 60 kph uphill. It takes Emmy 6 hours to get from Functionville to Alegbraton, and 4 hours to return. How far is it between the two towns?



ANSWER: We believe this puzzle is originally due to Lewis Carroll. Suppose the trip from Functionville to Algebraton is A km downhill, B km level and C km uphill. Then

$$A/120 + B/80 + C/60 = 6$$

The reverse trip gives us that

$$A/60 + B/80 + C/120 = 4$$

Multiplying both of these equations by 240 and adding, we find that

$$6A + 6B + 6C = 2400$$

That is, the distance $A + B + C$ between the two towns is 400 km.